Homological quantum error correcting codes and real projective space

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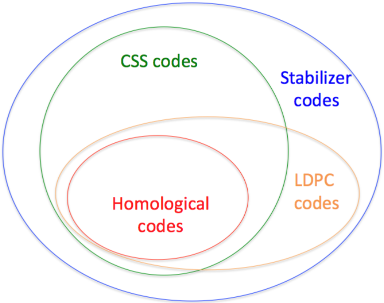
Outline

Quantum error correcting codes

- qubit: $\alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.
- two types of errors: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- $[\![n, k, d]\!]$ error correcting code: k logical qubits but n physical qubits (n $>$ k)
- minimal distance d of the code proportional to the maximal number of errors which can be corrected.

Families of codes

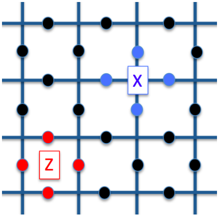
- stabilizer codes: the codespace is the 1-eigenspace of a set of commuting operators.
- CSS codes: can be specified by two orthogonal linear classical codes $\mathcal{C}_X$ and $\mathcal{C}_Z$.
- LDPC codes: lines and columns of parity check matrices have bounded weights.



- homological codes: The two orthogonal classical codes are defined from a cellulation of a manifold.

Example of homological code: Toric code [Kitaev 2002]

- cellulation of the torus by squares.
- To each edge corresponds a physical bit.
- To each face corresponds a line of H_Z.
- To each vertex corresponds a line of H_X.

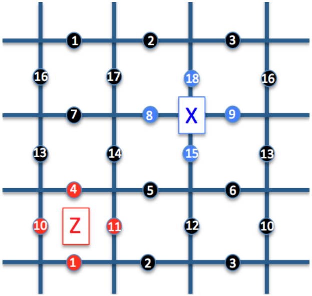


A torus is obtained by identifying left and right sides and identifying up and down sides of the square.

$\mathcal{C}_X$ and $\mathcal{C}_Z$ are orthogonal because each (face,vertex) pair shares an even number of edges.

Example of homological code: Toric code [Kitaev 2002]

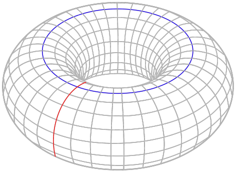
- The line of H_Z corresponding to the red face checks the parity of the bits 1, 10, 4 and 11.
- The line of H_X corresponding to the blue vertex checks the parity of the bits 8, 18, 9 and 15.



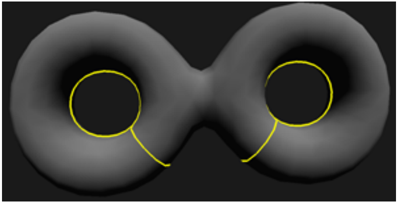
$\mathcal{C}_X$ and $\mathcal{C}_Z$ are orthogonal because each (face,vertex) pair shares an even number of edges.

Geometric interpretation of k

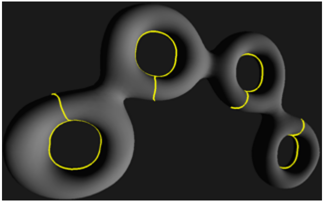
- The code dimension is the rank of the first homology group H_1 of the manifold.
- Informally, it is the number of different loops of the manifold.



$k = 2 = rank(H_1)$



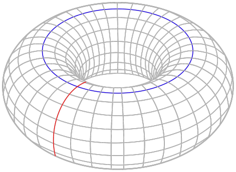
$k = 4 = rank(H_1)$



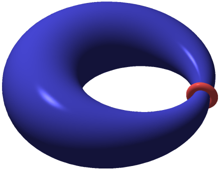
$k = 8 = rank(H_1)$

Geometric interpretation of n and d

- the code length n is proportional to the area of the manifold.
- the minimal distance d is proportional to the systole of the manifold.
- the systole is the length of the shortest non contractible loop of the manifold.



$$d = systole \approx 20$$
$$n = 2 \times area \approx 800$$

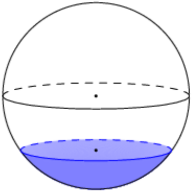


d = systole = length of red circle

positive curvature and the sphere S^2

goal: family of codes satisfying $n = o(d^2)$

$$
\begin{aligned}
area(disk_{euclidean}(r)) &= \pi r^2 \\
area(disk_{spherical}(r)) &= 2\pi(1 - cos(r)) \\
&= \pi r^2 - \frac{\pi}{12} r^4 + O(r^6)
\end{aligned}
$$

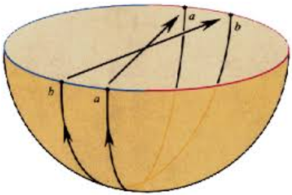


$area(disk_{spherical}) < area(disk_{euclidean})$

But every loop on the sphere is contractible to a point.
Hence the dimension k of a code defined on the sphere is zero.

The real projective plane

Identify every pair of antipodal points.
Some loops cannot be contracted to a point.

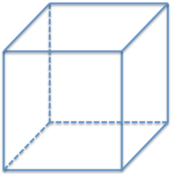


- *systole*$_{\text{projective plane}} = \pi$
- *area*$_{\text{projective plane}} = 2\pi$
  - $(\textit{systole}_{\text{projective plane}})^2 > \textit{area}_{\text{projective plane}}$

But the area of a real projective plane of constant curvature $+1$ is bounded above by 2π.
A solution is to increase dimension.

A discrete model of the real projective plane

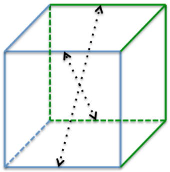
Identify pair of antipodal points of the cube.



A discrete model of the real projective plane

Identify pair of antipodal points of the cube.

- $n = 6$



opposite edges are identified

A discrete model of the real projective plane

$$H_X = mat_{adjacency}(vertices, edges)$$

Identify pair of antipodal points of the cube.

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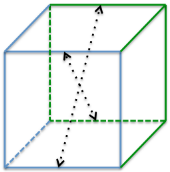
- $d_X = 3$



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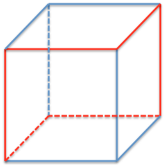


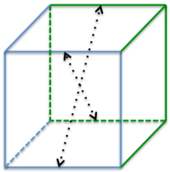
least weight codeword for the code $\mathcal{C}_X$

A discrete model of the real projective plane

$$H_Z = mat_{adjacency}(faces, edges)$$

Identify pair of antipodal points of the cube.

- n $= 6$
- $d_X = 3$
- $d_Z = 2$
- $d = min(d_X, d_Z) = 2$



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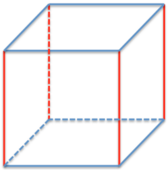

least weight codeword for the code $\mathcal{C}_Z$

A discrete model of the real projective 2m-space

Identify pair of antipodal points of the hypercube of dimension 2m+1.
Bits are identified with m-faces of the hypercube.

With the 5-hypercube:

- $n = 40$
- $d_X = 10$
- $d_Z = 4$
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- $d_X = \binom{2m+1}{m}$
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  Remark: $n = d_X d_Z$

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Comparison with toric codes

- High dimensional projective code:
  - $d \leq \sqrt{n}$
  - By identifying bits with ℓ-faces of the hypercube, $\ell > m$, one could reach $d = \sqrt{n}$.
  - logarithmically LDPC.
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Conclusions

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- High dimensional manifolds lead to smaller minimal distances.
- Homological quantum error correcting codes can be understood geometrically.

The only known family of codes satisfying $d = \omega(\sqrt{n})$ is a homological code built on a four dimensional manifold with non zero curvature [Freedman 2002].

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Thank you for your attention! Questions?